

# POLYNOMIAL AND MULTILINEAR HARDY–LITTLEWOOD INEQUALITIES: ANALYTICAL AND NUMERICAL APPROACHES

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**ABSTRACT.** We investigate the growth of the polynomial and multilinear Hardy–Littlewood inequalities. Analytical and numerical approaches are performed and, in particular, among other results, we show that a simple application of the best known constants of the Clarkson inequality improves a recent result of Araujo et al. We also obtain the optimal constants of the generalized Hardy–Littlewood inequality in some special cases.

## 1. INTRODUCTION

The investigation of polynomials and multilinear operators acting on Banach spaces is a fruitful topic of investigation that dates back to the 30's (see, for instance [10, 19, 20] and, for recent papers, [7, 8, 11, 15, 17] among many others).

Let  $\mathbb{K}$  be the real or complex scalar field, and  $n \geq 1$  be a positive integer. In 1930 Littlewood proved his well-known  $4/3$  inequality to solve a problem posed by P.J. Daniell (see [20]). The Littlewood's  $4/3$  inequality asserts that

$$\left( \sum_{i,j=1}^n |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|T\|$$

for all positive integers  $n$  and every continuous bilinear form  $T : c_0 \times c_0 \rightarrow \mathbb{K}$ , where  $\|T\| := \sup_{z^{(1)}, z^{(2)} \in B_{c_0}} |T(z^{(1)}, z^{(2)})|$ . The exponent  $4/3$  is optimal and in the case  $\mathbb{K} = \mathbb{R}$  the optimality of the constant  $\sqrt{2}$  is also known (see [16]). Soon afterwards this inequality was generalized by Hardy and Littlewood ([19], 1934) for bilinear forms on  $\ell_p$  and, in 1982 Praciano-Pereira ([27]) extended the result of Hardy and Littlewood to  $m$ -linear forms on  $\ell_p$ . Another generalization of the Hardy–Littlewood inequalities for  $m$ -linear forms was obtained by Dimant and Sevilla-Peris, and will be treated in Remark 2.3.

The Hardy–Littlewood inequalities for  $m$ -linear forms is the following result:

**Theorem (Hardy–Littlewood/Praciano-Pereira).** Let  $m \geq 2$  be a positive integer. For  $p \geq 2m$ , there is a constant  $C_{\mathbb{K},m,p} \geq 1$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^n |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p} \|T\|,$$

for all positive integers  $n$  and all  $m$ -linear forms  $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ .

The exponent  $\frac{2mp}{mp+p-2m}$  is optimal and  $\|T\| := \sup_{z^{(1)}, \dots, z^{(m)} \in B_{\ell_p^n}} |T(z^{(1)}, \dots, z^{(m)})|$ . In the limiting case ( $p = \infty$ , considering, of course  $f(\infty) := \lim_{p \rightarrow \infty} f(p)$  regardless of the function  $f$ ), we recover the classical multilinear Bohnenblust–Hille inequality (see [10]). The original upper estimate for  $C_{\mathbb{K},m,p}$  is  $2^{\frac{m-1}{2}}$ . Recently, in some papers (see [5, 6, 24]), this estimate was improved for all  $m$  and  $p$  with the only exception of the case  $C_{\mathbb{R},m,2m}$ .

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The precise behavior of the growth of the optimal constants  $C_{\mathbb{K},m,p}$  is still unknown (some partial results can be found in [4, 5, 6]).

Up to now, the best known lower estimates for  $C_{\mathbb{R},m,p}$  are always smaller than 2 and again the more critical situation is when  $p = 2m$ , where the lower estimates presented in [4] are more difficult to obtain and not explicitly stated for the case  $p = 2m$ .

In view of the special role played by the constants  $C_{\mathbb{R},m,2m}$  and since this case is a kind of dual version of the classical Bohnenblust–Hille inequality (see details in Section 2), in the Sections 3 and 4 we investigate this critical case and obtain quite better lower estimates. Our approach has two novelties: a new class of multilinear forms, not investigated before in similar context, and a new numerical approach in this framework. As it will be clear along the paper the new family of multilinear forms introduced in this paper is more effective to obtain good lower estimates for the Hardy–Littlewood inequality.

In Section 5 we investigate the generalized Hardy–Littlewood inequality. Our approach provides new lower bounds for this inequality. As a consequence of our results, in Theorem 5.2 we obtain optimal constants for some cases of three-linear forms.

In Section 6 we investigate the polynomial Hardy–Littlewood inequality. The approaches of Sections 5 and 6 are entirely analytic and do not depend on computation assistance.

## 2. THE MULTILINEAR HARDY–LITTLEWOOD INEQUALITY

From now on, if  $p \in (1, 2)$ ,  $p^*$  is the extended real number such that  $\frac{1}{p} + \frac{1}{p^*} = 1$ . Also,  $E'$  denotes the topological dual of a Banach space  $E$ . By  $\mathcal{L}(^m E; F)$  we denote the Banach space of all (bounded)  $m$ -linear operators  $U : E \times \cdots \times E \rightarrow F$ , with  $E, F$  Banach spaces over  $\mathbb{K}$ . For  $1 \leq s \leq r < \infty$ ,  $U \in \mathcal{L}(^m E; F)$  is called *multiple  $(r, s)$ -summing*, if there exists a constant  $C > 0$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^n \|U(x_{i_1}, \dots, x_{i_m})\|_F^r \right)^{\frac{1}{r}} \leq C \|U\| \prod_{k=1}^m \|(x_{i_k})_{i_k=1}^n\|_{w,s}$$

for all finite choice of vectors  $x_{i_k} \in E$ ,  $1 \leq i_k \leq n$ ,  $1 \leq k \leq m$ , where

$$\|(x_i)_{i=1}^n\|_{w,s} := \sup_{\|\varphi\|_{E'} \leq 1} \left( \sum_{i=1}^n |\varphi(x_i)|^s \right)^{\frac{1}{s}}.$$

The vector space of all multiple  $(r, s)$ -summing operators in  $\mathcal{L}(^m E; F)$  is denoted by  $\Pi_{(r,s)}(^m E; F)$ . For more details of the theory of multiple summing operators theory see [22, 25, 26].

In the terminology of the multiple summing operators, it is well known (see, for instance, [15, Section 5]) that the Hardy–Littlewood/Praciano-Pereira inequality is equivalent to the equality

$$\Pi_{(\frac{2mp}{mp+p-2m}; p^*)}(^m E; \mathbb{K}) = \mathcal{L}(^m E; \mathbb{K}).$$

In other words, if  $m \geq 2$  and  $p \geq 2m$ , then there is a constant  $C_{\mathbb{K},m,p} \geq 1$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^n |T(x_{i_1}, \dots, x_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p} \|T\| \prod_{k=1}^m \|(x_{i_k})_{i_k=1}^n\|_{w,p^*}$$

for all  $m$ -linear forms  $T : E \times \cdots \times E \rightarrow \mathbb{K}$ , for all finite choice of vectors  $x_{i_k} \in E$ ,  $1 \leq i_k \leq n$ ,  $1 \leq k \leq m$ .

As mentioned in the introduction, the case  $p = 2m$  in the Hardy–Littlewood inequality is specially interesting. In this case we have very few information on the constants involved, and moreover, this case is a kind of dual version of the Bohnenblust–Hille inequality, in the sense that in the pair of parameters  $(\frac{2mp}{mp+p-2m}; p^*)$ , each case has a coordinate which is kept constant (in reverse location). More specifically, in the terminology of the multiple summing operators, the Bohnenblust–Hille inequality asserts that

$$\Pi_{(\frac{2m}{m+1}; 1)}(^m E; \mathbb{K}) = \mathcal{L}(^m E; \mathbb{K})$$

for all Banach spaces  $E$ . On the other hand, when  $p = 2m$ , the Hardy–Littlewood inequality is equivalent to

$$\Pi_{(2; \frac{2m}{2m-1})}(^m E; \mathbb{K}) = \mathcal{L}(^m E; \mathbb{K})$$

for all Banach spaces  $E$ .

Up to now the best known upper estimates for the constants  $(C_{\mathbb{R},m,p})_{m=1}^{\infty}$  can be found in [6, page 1887] and [24]. The updated results on the lower bounds for these constants are:

- $C_{\mathbb{R},m,p} \geq 2^{\frac{mp+2m-2m^2-p}{mp}}$  for  $p > 2m$  and  $C_{\mathbb{R},m,p} > 1$  for  $p = 2m$  (see [4]);

From now on  $p^*$  denotes the conjugate number of  $p$ . In this section we find an overlooked (and simple) connection between the Clarkson's inequalities and the Hardy–Littlewood's constants which helps to find analytical lower estimates (without the use of a computational aid) for these constants.

**Theorem 2.1.** *Let  $m \geq 2$  and  $p \geq 2m$ . The optimal constants of the Hardy–Littlewood inequalities satisfies*

$$C_{\mathbb{R},m,p} \geq \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{\sup_{x \in [0,1]} \frac{((1+x)^{p^*} + (1-x)^{p^*})^{\frac{1}{p^*}}}{(1+x^p)^{1/p}}}.$$

*Proof.* For a given Banach space  $E$  we know that  $\Psi : \mathcal{L}(^2 E; \mathbb{R}) \rightarrow \mathcal{L}(E; E^*)$  given by  $\Psi(T)(x)(y) = T(x, y)$  is an isometric isomorphism. For  $E = \ell_p^2$  and using the characterization of the dual of  $\ell_p^2$ , we conclude that for the bilinear form

$$\begin{aligned} T_{2,p} : \quad \ell_p^2 \times \ell_p^2 &\rightarrow \mathbb{R} \\ ((x_i^{(1)}), (x_i^{(2)})) &\mapsto x_1^{(1)} x_1^{(2)} + x_1^{(1)} x_2^{(2)} + x_2^{(1)} x_1^{(2)} - x_2^{(1)} x_2^{(2)}, \end{aligned}$$

we have

$$\begin{aligned} \Psi(T_{2,p}) : \quad \ell_p^2 &\rightarrow \ell_{p^*}^2 \\ (x_i) &\mapsto (x_1 + x_2, x_1 - x_2). \end{aligned}$$

Since  $p \geq 2m$  and  $m \geq 2$ , using the best constants from the Clarkson's inequality in the real case (see [21, Theorem 2.1]) we know the norm of the linear operator  $\Psi(T_{2,p})$  (and consequently the norm of the bilinear form  $T_{2,p}$ ), i.e.,

$$\|T_{2,p}\| = \|\Psi(T_{2,p})\| = \sup_{x \in [0,1]} \frac{((1+x)^{p^*} + (1-x)^{p^*})^{\frac{1}{p^*}}}{(1+x^p)^{1/p}}.$$

Now, as in [4], we define inductively

$$\begin{aligned} T_{m,p} : \quad \ell_p^{2^{m-1}} \times \dots \times \ell_p^{2^{m-1}} &\rightarrow \mathbb{R} \\ (x^{(1)}, \dots, x^{(m)}) &\mapsto (x_1^{(m)} + x_2^{(m)})T_{m-1,p}(x^{(1)}, \dots, x^{(m)}) \\ &\quad + (x_1^{(m)} - x_2^{(m)})T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), \dots, B^{2^{m-1}}(x^{(m-1)})), \end{aligned}$$

where  $x^{(k)} = (x_j^{(k)})_{j=1}^{2^{m-1}} \in \ell_p^{2^{m-1}}$ ,  $1 \leq k \leq m$ , and  $B$  is the backward shift operator in  $\ell_p^{2^{m-1}}$  and, again as in [4], we conclude that

$$\begin{aligned} |T_{m,p}(x^{(1)}, \dots, x^{(m)})| &\leq |x_1^{(m)} + x_2^{(m)}| |T_{m-1,p}(x^{(1)}, \dots, x^{(m)})| \\ &\quad + |x_1^{(m)} - x_2^{(m)}| |T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), B^{2^{m-2}}(x^{(2)}), \dots, B^{2^{m-1}}(x^{(m-1)}))| \\ &\leq \|T_{m-1,p}\| (|x_1^{(m)} + x_2^{(m)}| + |x_1^{(m)} - x_2^{(m)}|) \\ &\leq 2\|T_{m-1,p}\| \|x^{(m)}\|_p, \end{aligned}$$

i.e.,

$$\|T_{m,p}\| \leq 2^{m-2} \|T_{2,p}\|.$$

□

Now we have

$$(4^{m-1})^{\frac{mp+p-2m}{2mp}} = \left( \sum_{j_1, \dots, j_m=1}^{2^{m-1}} |T_{m,p}(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{R},m,p} 2^{m-2} \|T_{2,p}\|$$

and thus

$$C_{\mathbb{R},m,p} \geq \frac{(4^{m-1})^{\frac{mp+p-2m}{2mp}}}{2^{m-2} \|T_{2,p}\|} = \frac{2^{2(m-1)(\frac{mp+p-2m}{2mp}) - (m-2)}}{\sup_{x \in [0,1]} \frac{((1+x)^{p^*} + (1-x)^{p^*})^{1/p^*}}{(1+x^p)^{1/p}}}$$

When  $m = 2$ , using estimates of [21, page 1369], note that

$$\begin{aligned} C_{\mathbb{R},2,4} &\geq \frac{2}{\sqrt{3}} > 1.1546 \\ C_{\mathbb{R},2,8} &\geq \frac{2^{\frac{5}{4}}}{1.892} > 1.2570 \\ C_{\mathbb{R},2,p} &\geq \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{1.9836} > 1.3591 \text{ for } p = 1 + \log_{9/10} 1/19 \\ C_{\mathbb{R},2,p} &\geq \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{1.9999} > 1.4105 \text{ for } p = 1 + \log_{99/100} 1/199. \end{aligned}$$

Using the old estimates of [4] for  $p > 2m$  (i.e.,  $C_{\mathbb{R},m,p} \geq 2^{\frac{mp+2m-2m^2-p}{mp}}$ ) we can easily verify that the old estimates are worse. Also, in the old estimates we have no closed formula for the case  $p = 2m$ .

**Remark 2.2.** *One may try to use the complex Clarkson's inequalities to obtain nontrivial lower bounds for the constants of the complex Hardy-Littlewood inequality. But, this is not effective, since we just get trivial lower bounds, i.e., 1.*

**Remark 2.3** (The case  $m < p < 2m$ ). *There is also a version of Hardy-Littlewood's inequality for  $m < p < 2m$ , due to Dimant and Sevilla-Peris ([15] and the forthcoming Section 6). In this case, the optimal exponent is  $\frac{p}{p-m}$  and we still denote the optimal constant for this inequality by  $C_{\mathbb{K},m,p}$ . The best information we have so far for the lower estimates for the constant  $C_{\mathbb{R},m,p}$  are trivial, that is,*

$$1 \leq C_{\mathbb{R},m,p} \leq (\sqrt{2})^{m-1}.$$

*Similarly to the argument used in the proof of the Theorem 2.1, we can also provide a closed formula (which depends on  $p$ ) for the lower bounds of  $C_{\mathbb{R},m,p}$ , but in this case, we do not always have nontrivial information. More precisely, we prove that*

$$C_{\mathbb{R},m,p} \geq \frac{2^{\frac{mp+2m-2m^2}{p}}}{\sup_{x \in [0,1]} \frac{((1+x)^{p^*} + (1-x)^{p^*})^{\frac{1}{p^*}}}{(1+x^p)^{\frac{1}{p}}}}.$$

*It is important to mention this case because, for suitable choices of  $p$ , we get nontrivial lower estimates for  $C_{\mathbb{R},m,p}$ . For instance,*

$$C_{\mathbb{R},2,7/2} \geq 1.104, \quad C_{\mathbb{R},3,28/5} \geq 1.025, \quad \text{and} \quad C_{\mathbb{R},100,199999/1000} \geq 1.003.$$

*This leads us to question the following: Would also be the optimal constants of the Hardy-Littlewood inequality for  $m < p < 2m$  strictly greater than 1?*

### 3. FIRST NUMERICAL ESTIMATES (USING WELL-KNOWN MULTILINEAR FORMS)

Since the publication of [16], the family of  $m$ -linear forms  $T_m : \ell_\infty \times \cdots \times \ell_\infty$  defined inductively by

$$(3.1) \quad T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2,$$

$$(3.2) \quad \begin{aligned} T_3(x, y, z) &= (z_1 + z_2)(x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2) \\ &\quad + (z_1 - z_2)(x_3 y_3 + x_3 y_4 + x_4 y_3 - x_4 y_4), \end{aligned}$$

$$(3.3) \quad \begin{aligned} T_4(x, y, z, w) &= (w_1 + w_2) \begin{pmatrix} (z_1 + z_2)(x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2) \\ + (z_1 - z_2)(x_3 y_3 + x_3 y_4 + x_4 y_3 - x_4 y_4) \end{pmatrix} \\ &\quad + (w_1 - w_2) \begin{pmatrix} (z_3 + z_4)(x_5 y_5 + x_5 y_6 + x_6 y_5 - x_6 y_6) \\ + (z_3 - z_4)(x_7 y_7 + x_7 y_8 + x_8 y_7 - x_8 y_8) \end{pmatrix} \end{aligned}$$

and so on, have been used to find lower estimates for Bohnenblust-Hille and related inequalities (see also [24]). In the context of the Hardy-Littlewood inequalities we also have good results, but in the next section we invent different multilinear forms that, in our context, provide better estimates.

The numerical issue involved to obtain our estimates is the calculus of  $\|T_m\|$  when  $\ell_\infty$  is replaced by  $\ell_p$  (in this case we write  $T_{m,p}$  instead of  $T_m$ ). This task refers to a typical nonlinear optimization

problem subject to restrictions. Namely, we want to find a global maximum of  $|T_{m,2m}(x^{(1)}, \dots, x^{(m)})|$  with  $x^{(i)} \in B_{\ell_{2m}}$ ,  $i = 1, \dots, m$  for the operators (3.1), (3.2), (3.3), etc.

To perform this computer-aided calculus we use a couple of software: multi-paradigm numerical computing environment called MATLAB (MATrix LABoratory) (see [18]) to specify the problem and a software library for large-scale nonlinear optimization called Interior Point to solve it. Mathematical details of the algorithm used by interior-point can be found in several publications (see for instance [12, 13, 28]).

As the interior-point algorithm is designed to find local solutions for a given optimization problem starting from a initial data, it is necessary to find all local solutions (all maxima) and take the greatest of them. This can be done taking a reasonable distribution of starting points throughout the domain of the operator.

Performing these calculations for  $T_{m,2m}$ , we obtain

$$(3.4) \quad \begin{array}{|c|c|c|} \hline C_{\mathbb{R},2,4} > & \frac{2}{1.74} & > 1.149 \\ \hline C_{\mathbb{R},3,6} > & \frac{4}{3.29} & > 1.215 \\ \hline C_{\mathbb{R},4,8} > & \frac{8}{6.40} & > 1.250 \\ \hline C_{\mathbb{R},5,10} > & \frac{16}{12.60} & > 1.269 \\ \hline C_{\mathbb{R},6,12} > & \frac{32}{25.00} & > 1.280 \\ \hline C_{\mathbb{R},7,14} > & \frac{64}{49.47} & > 1.293 \\ \hline C_{\mathbb{R},8,16} > & \frac{128}{98.36} & > 1.301 \\ \hline C_{\mathbb{R},9,18} > & \frac{256}{195.81} & > 1.307. \\ \hline \end{array}$$

#### 4. NEW MULTILINEAR FORMS AND BETTER ESTIMATES

Up to now the best known multilinear forms to use in order to find lower bounds for the Bohnenblust–Hille and Hardy–Littlewood inequalities were those defined in (3.1), (3.2), (3.3) and so on. Now we show that for  $m = 4, 8, 16, \dots$  we get better estimates using slightly different multilinear forms and numerical computation. Define

$$\tilde{T}_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2,$$

$$\begin{aligned} \tilde{T}_4(x, y, z, w) &= (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2)(z_1w_1 + z_1w_2 + z_2w_1 - z_2w_2) \\ &\quad + (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2)(z_3w_3 + z_3w_4 + z_4w_3 - z_4w_4) \\ &\quad + (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)(z_1w_1 + z_1w_2 + z_2w_1 - z_2w_2) \\ &\quad - (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)(z_3w_3 + z_3w_4 + z_4w_3 - z_4w_4), \end{aligned}$$

$$\begin{aligned} \tilde{T}_8(x, y, z, w, r, s, t, u) &= \tilde{T}_4(x, y, z, w)\tilde{T}_4(r, s, t, u) \\ &\quad + \tilde{T}_4(x, y, z, w)\tilde{T}_4(B^4(r), B^4(s), B^4(t), B^4(u)) \\ &\quad + \tilde{T}_4(B^4(x), B^4(y), B^4(z), B^4(w))\tilde{T}_4(r, s, t, u) \\ &\quad - \tilde{T}_4(B^4(x), B^4(y), B^4(z), B^4(w))\tilde{T}_4(B^4(r), B^4(s), B^4(t), B^4(u)), \end{aligned}$$

and so on (recall that  $B^4$  is the shift operator, as defined before). Using  $\tilde{T}_4, \tilde{T}_8$ , etc, we obtain

$$(4.1) \quad \begin{array}{|c|c|c|} \hline C_{\mathbb{R},4,8} > & \frac{2^3}{6.20} & > 1.290 \\ \hline C_{\mathbb{R},8,16} > & \frac{2^7}{91.48} & > 1.399 \\ \hline C_{\mathbb{R},16,32} > & \frac{2^{15}}{22137.70} & > 1.480, \\ \hline \end{array}$$

and this procedure seems clearly better than the former.

## 5. ON THE GENERALIZED HARDY–LITTLEWOOD INEQUALITY

The main goal in this section is to provide optimal constants for some cases of three-linear forms in the recently extended version of the Hardy–Littlewood inequality, presented in [2, 15]:

**Theorem (Generalized Hardy–Littlewood inequality).** If  $m \geq 2$  is a positive integer,  $2m \leq p \leq \infty$  and  $\mathbf{q} := (q_1, \dots, q_m) \in \left[\frac{p}{p-m}, 2\right]^m$  are such that

$$(5.1) \quad \frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p},$$

then there exists a constant  $C_{m,p,\mathbf{q}}^{\mathbb{K}} \geq 1$  such that

$$(5.2) \quad \left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \left( \dots \left( \sum_{j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{m,p,\mathbf{q}}^{\mathbb{K}} \|T\|$$

for all continuous  $m$ -linear forms  $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$  and all positive integers  $n$ .

The case  $p = \infty$  recovers the so called generalized Bohnenblust–Hille inequality (see [1]) and when  $p = \infty$  and  $q_1 = \dots = q_m = \frac{2m}{m+1}$  we recover the classical Bohnenblust–Hille inequality. The optimal constants  $C_{m,p,\mathbf{q}}^{\mathbb{K}}$  are known in very few cases, namely

(i)  $p = \infty$  and  $(q_1, \dots, q_m) = (1, 2, \dots, 2)$ . In this case (see [24, Theorem 2.1])  $C_{m,\infty,\mathbf{q}}^{\mathbb{R}} = (\sqrt{2})^{m-1}$  for all  $m \geq 2$ ;

(ii)  $(m, p) = (2, \infty)$  and no restriction on  $q_1, q_2$ . In this case (see [1, Theorem 6.3])  $C_{2,\infty,\mathbf{q}}^{\mathbb{R}} = \sqrt{2}$ .

In these two cases these optimal constants are obtained by using special multilinear forms to find lower bounds that match exactly with the known upper bounds of the constants. This approach seems to be not effective in other cases, but we do not know if the reason is a fault of the method or a weakness of our estimates of upper bounds (i.e, maybe the known upper bounds are not good enough). Using this technique it was recently shown in [24, Theorem 2.3] that for a constant  $\alpha \in [1, 2]$  and a multiple exponent  $\mathbf{q} = (\alpha, \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}, \dots, \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha})$ , we have

$$(5.3) \quad C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{\frac{2m - \alpha m - 4 + 3\alpha}{2\alpha}}.$$

By using the Minkowski inequality, we obtain that for  $\mathbf{q} = (\frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}, \dots, \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}, \alpha)$ , with  $\alpha > \frac{2m}{m+1}$  the estimate (5.3) gives us

$$C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{\frac{2m - \alpha m - 4 + 3\alpha}{2\alpha}}.$$

In this section, we show that for a constant  $\alpha \in [1, 2]$  and a  $\mathbf{q} = (\frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}, \dots, \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}, \alpha)$  we have

$$(5.4) \quad C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{\frac{3\alpha m - 2m - 5\alpha + 4}{2\alpha(m-1)}}.$$

For  $\alpha > \frac{2m}{m+1}$  the new estimate is strictly bigger (and thus better) than the previous.

When  $m = 3$  and  $\alpha = 2$  we obtain  $C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{3/4}$  and since we already know (see [5, Lemma 2.1]) that  $C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \leq 2^{3/4}$ , we conclude that the optimal constant is  $2^{3/4}$ .

The result proved here is:

**Proposition 5.1.** *Let  $\alpha \in [1, 2]$  be a constant and  $\mathbf{q} = (\beta_m, \dots, \beta_m, \alpha)$  be a multiple exponent of the generalized Bohnenblust–Hille inequality for real scalars. Then*

$$C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{\frac{3\alpha m - 2m - 5\alpha + 4}{2\alpha(m-1)}}.$$

*Proof.* The  $m$ -linear operators that we will use are defined inductively as in (3.1), (3.2) and (3.3). Since

$$\begin{aligned} \left( \frac{(2^{m-1})^2}{2} 2^{\frac{1}{\alpha}\beta_m} \right)^{\frac{1}{\beta_m}} &= \left( 2^{2m-3} 2^{\frac{1}{\alpha}\beta_m} \right)^{\frac{1}{\beta_m}} = \left( \sum_{i_1, \dots, i_{m-1}=1}^{2^{m-1}} \left( \sum_{i_m=1}^2 |T_m(e_{i_1}, \dots, e_{i_m})|^\alpha \right)^{\frac{1}{\alpha}\beta_m} \right)^{\frac{1}{\beta_m}} \\ &\leq C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \|T_m\| \end{aligned}$$

and  $\beta_m = \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}$  we conclude that

$$C_{m,\infty,\mathbf{q}}^{\mathbb{R}} \geq \frac{\left( 2^{2m-3} (2)^{\frac{1}{\alpha}\beta_m} \right)^{\frac{1}{\beta_m}}}{2^{m-1}} = 2^{\frac{3\alpha m - 2m - 5\alpha + 4}{2\alpha(m-1)}}.$$

□

**Theorem 5.2.** *The optimal constant of the generalized Bohnenblust–Hille inequality for  $m = 3$  and  $\mathbf{q} = (4/3, 4/3, 2)$  or  $\mathbf{q} = (4/3, 8/5, 8/5)$  or  $\mathbf{q} = (4/3, 2, 4/3)$  is  $C_{3,\infty,\mathbf{q}}^{\mathbb{R}} = 2^{3/4}$ .*

*Proof.* From [5, Lemma 2.1] we obtain, for  $m$  and  $\mathbf{q}$  satisfying the hypotheses of the theorem, the estimate

$$C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \leq 2^{3/4}.$$

Using (5.4) we prove that for  $\mathbf{q} = (4/3, 4/3, 2)$  we have

$$C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{3/4}$$

and, finally, using (5.3) we show that for  $\mathbf{q} = (4/3, 8/5, 8/5)$  we have

$$C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \geq 2^{3/4}.$$

On the other hand, using the operator  $T_3$  (see (3.2)) we have

$$2^{\frac{11}{4}} = \left( \sum_{i_1=1}^4 \left( \sum_{i_2=1}^4 \left( \sum_{i_3=1}^2 |T_3(e_{i_1}, e_{i_2}, e_{i_3})|^{4/3} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \right)^{\frac{3}{4}},$$

and thus for  $\mathbf{q} = (4/3, 2, 4/3)$  we get

$$C_{3,\infty,\mathbf{q}}^{\mathbb{R}} \geq \frac{2^{\frac{11}{4}}}{2^2} = 2^{\frac{3}{4}}.$$

□

## 6. THE POLYNOMIAL HARDY–LITTLEWOOD INEQUALITY

Let  $E$  be a real or complex Banach space and  $m$  be a positive integer and let  $\mathbb{K}$  be the real or complex scalar field. A map  $P : E \rightarrow \mathbb{K}$  is a homogeneous polynomial on  $E$  of degree  $m$  if there exists a symmetric  $m$ -linear form  $L$  on  $E^m$  such that  $P(x) = L(x, \dots, x)$  for all  $x \in E$ . We denote by  $\mathcal{P}^m(E)$  the space of continuous  $m$ -homogeneous polynomials on  $E$  endowed with the usual norm

$$\|P\| := \sup\{|P(x)| : \|x\| = 1\}.$$

Observe that an  $m$ -homogeneous polynomial in  $\mathbb{K}^n$  can be written as

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha,$$

where  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . We denote

$$|P|_p := \left( \sum_{|\alpha|=m} |a_\alpha|^p \right)^{1/p}$$

and

$$|P|_\infty := \max |a_\alpha|.$$

The polynomial Hardy–Littlewood inequality is:

**Theorem (Polynomial Hardy–Littlewood inequality).** For  $m < p \leq \infty$  there is a constant  $D_{\mathbb{K},m,p} \geq 1$  such that

$$(6.1) \quad \begin{aligned} \left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} &\leq D_{\mathbb{K},m,p} \|P\|, \quad \text{if } m < p \leq 2m, \\ \left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} &\leq D_{\mathbb{K},m,p} \|P\|, \quad \text{if } p \geq 2m \end{aligned}$$

for all positive integers  $n$  and all  $m$ -homogeneous polynomials  $P : \ell_p^n \rightarrow \mathbb{K}$  given by

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha.$$

This is a consequence of the multilinear Hardy–Littlewood inequality, previously described, and the following inequality also known as Hardy–Littlewood inequality [15]:

**Theorem (Hardy–Littlewood/Dimant–Sevilla-Peris).** For  $m < p \leq 2m$ , there is a constant  $C_{\mathbb{K},m,p} \geq 1$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^n |T(e_{i_1}, \dots, e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{\mathbb{K},m,p} \|T\|$$

for all positive integers  $n$  and all  $m$ -linear forms  $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ .

Above, the exponent  $\frac{p}{p-m}$  is optimal and therefore in (6.1) both exponents  $\frac{p}{p-m}$  and  $\frac{2mp}{mp+p-2m}$  are optimal. The case  $p = \infty$  in the appropriate inequality of (6.1), is the classical polynomial Bohnenblust–Hille inequality (see [10]).

From now on  $D_{\mathbb{K},m,p}$  denotes the optimal constants satisfying (6.1). As in the multilinear case, the precise behaviour of the growth of the constants  $D_{\mathbb{K},m,p}$  is still unknown (partial results can be found in [3, 23]). For instance, in [3, Theorem 3.1] it is proved that for  $p \geq 2m$  we have

$$D_{\mathbb{R},m,p} \geq \left( \sqrt[16]{2} \right)^m.$$

When  $p = \infty$  we know that (see [9, 14])

$$\begin{aligned} \limsup_m D_{\mathbb{R},m,\infty}^{1/m} &= 2; \\ \limsup_m D_{\mathbb{C},m,\infty}^{1/m} &= 1. \end{aligned}$$

It will be convenient to define  $H_1 = \{(p, m) \in \mathbb{R} \times \mathbb{N} : m < p < 2m\}$  and  $H_2 = \{(p, m) \in \mathbb{R} \times \mathbb{N} : p \geq 2m\}$  with any total order. The main results of this section are the following:

**Lemma 6.1.** *Let  $j = 1, 2$ . Then*

$$\limsup_{H_j} D_{\mathbb{R},m,p}^{1/m} \geq 2.$$

*Proof.* Consider the sequence of norm-one  $j$ -homogeneous polynomials  $Q_j : \ell_p \rightarrow \mathbb{R}$  defined recursively by

$$\begin{aligned} Q_2(x_1, x_2) &= x_1^2 - x_2^2, \\ Q_{2^m}(x_1, \dots, x_{2^m}) &= Q_{2^{m-1}}(x_1, \dots, x_{2^{m-1}})^2 - Q_{2^{m-1}}(x_{2^{m-1}+1}, \dots, x_{2^m})^2. \end{aligned}$$



From the proof of [14, Theorem 3.1], we known that

$$(6.2) \quad |Q_{2^m}^n|_\infty \geq \left( \frac{2^n}{n+1} \right)^{2^m-1}$$

for every natural number  $n, m$ .

Next, since for every homogeneous polynomial  $P$  we obviously have

$$|P|_p \geq |P|_\infty,$$

from (6.2) we conclude that

$$D_{\mathbb{R}, n2^m, p} \geq \left( \frac{2^n}{n+1} \right)^{2^m-1}.$$

Note that

$$D_{\mathbb{R}, n2^m, p}^{1/n2^m} \geq \left( \left( \frac{2^n}{n+1} \right)^{2^m-1} \right)^{\frac{1}{n2^m}} = \left( \frac{2^n}{n+1} \right)^{\frac{2^m-1}{n2^m}}$$

and making  $m \rightarrow \infty$  we have

$$\left( \frac{2^n}{n+1} \right)^{\frac{2^m-1}{n2^m}} \rightarrow \frac{2}{(n+1)^{1/n}}$$

and now making  $n \rightarrow \infty$  we have

$$\frac{2}{(n+1)^{1/n}} \rightarrow 2.$$

□

From now on we write

$$\begin{aligned} \rho(p, m) &= \frac{p}{p-m} \text{ if } m < p \leq 2m, \\ \rho(p, m) &= \frac{2mp}{mp+p-2m} \text{ if } p \geq 2m. \end{aligned}$$

Now we prove the theorem:

**Theorem 6.2.** *Let  $j = 1, 2$ . At least one of the following two sentences hold true:*

- (a)  $\limsup_{H_j} D_{\mathbb{R}, m, p}^{1/m} = 2$ .
- (b)  $\limsup_{H_j} D_{\mathbb{C}, m, p}^{1/m} > 1$ .

*Proof.* Suppose that (a) is not true for some  $j$ . So (using the previous result) we would have  $\limsup_{H_j} D_{\mathbb{R}, m, p}^{1/m} > (2 + \varepsilon) > 2$ . Therefore, for each  $k \in \mathbb{N}$  there is  $n_k \in \mathbb{N}$ ,  $(p_k, m_k) \in H_j$  and a  $m_k$ -homogeneous polynomial  $P_{m_k} : \ell_{p_k}^{n_k} \rightarrow \mathbb{R}$  such that

$$\left( \sum_{|\alpha|=m_k} |a_\alpha|^{\rho(p_k, m_k)} \right)^{\frac{1}{\rho(p_k, m_k)}} \leq D_{\mathbb{R}, m_k, p_k} \|P_{m_k}\|,$$

with

$$D_{\mathbb{R}, m_k, p_k} > (2 + \varepsilon)^{m_k}.$$

Considering the complexification of  $P_{m_k}$  we know that

$$\|(P_{m_k})_{\mathbb{C}}\| \leq 2^{m_k-1} \|P_{m_k}\|$$

and now looking for the complex polynomials  $(P_{m_k})_{\mathbb{C}}$  we would have

$$\begin{aligned} \left( \sum_{|\alpha|=m_k} |a_\alpha|^{\rho(p_k, m_k)} \right)^{\frac{1}{\rho(p_k, m_k)}} &\leq D_{\mathbb{C}, m_k, p_k} \|(P_{m_k})_{\mathbb{C}}\| \\ &\leq D_{\mathbb{C}, m_k, p_k} 2^{m_k-1} \|P_{m_k}\| \end{aligned}$$

and thus

$$D_{\mathbb{R}, m_k, p_k} \leq D_{\mathbb{C}, m_k, p_k} 2^{m_k - 1},$$

i.e.,

$$D_{\mathbb{R}, m_k, p_k}^{1/m_k} \leq D_{\mathbb{C}, m_k, p_k}^{1/m_k} 2^{\frac{m_k - 1}{m_k}} \leq 2 D_{\mathbb{C}, m_k, p_k}^{1/m_k}.$$

Now, since

$$D_{\mathbb{R}, m_k, p_k}^{1/m_k} > 2 + \varepsilon$$

we conclude that

$$D_{\mathbb{C}, m_k, p_k}^{1/m_k} > 1 + \frac{\varepsilon}{2} > 1$$

for all  $k$ , and thus

$$\limsup_{H_j} D_{\mathbb{C}, m, p}^{1/m} > 1.$$

Reciprocally, if (b) is not true for some  $j$ , then  $\limsup_{H_j} D_{\mathbb{C}, m, p}^{1/m} = 1$  and thus  $\limsup_{H_j} D_{\mathbb{R}, m, p}^{1/m} \leq 2$  and from the previous lemma we conclude that

$$\limsup_{H_j} D_{\mathbb{R}, m, p}^{1/m} = 2.$$

□

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